

THE TRIGONOMETRY OF ANGLES

- *from Pythagorus (circa 540 B.C.)
to Descartes (circa 1637)*

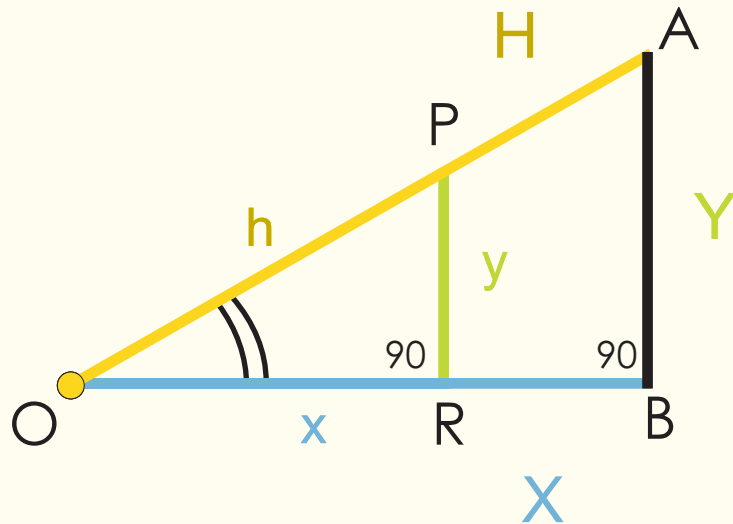
THE TRIGONOMETRY OF ANGLES

(triangle measurement of angles)

Recall from our tutorial on the trigonometric measurement of angles in the Euclidean Plane.

Similar Triangles

Figures are said to be similar if they have the same shape but not necessarily the same size.



— POR is similar to AOB

Corresponding sides of similar triangles are proportional.

$$\frac{H}{h} = \frac{Y}{y} = \frac{X}{x}$$

$$H = \overline{OA}$$

$$X = \overline{OB}$$

$$\frac{y}{h} = \frac{Y}{H}$$

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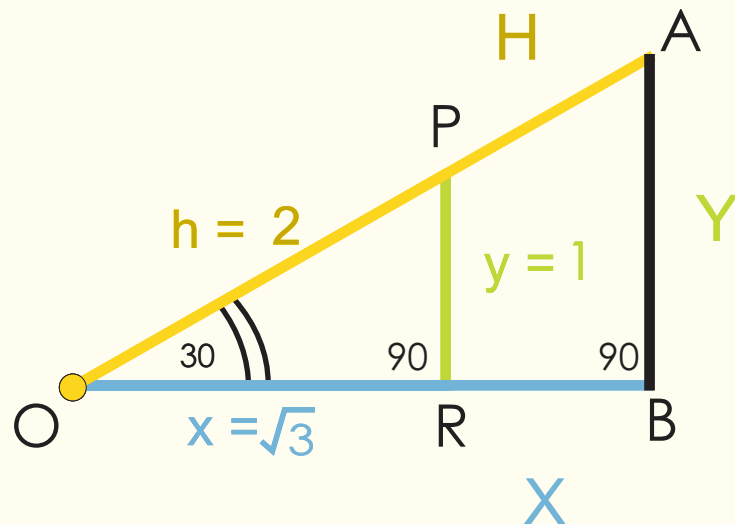
THE TRIGONOMETRY OF ANGLES

(triangle measurement of angles)

Let's review what we know about the trigonometric measures of an angle of 30 degrees.

Similar Triangles

Figures are said to be similar if they have the same shape but not necessarily the same size.



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$\triangle POR$ is similar to $\triangle AOB$

Corresponding sides of similar triangles are proportional.

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THE TRIGONOMETRY OF ANGLES

(triangle measurement of angles)

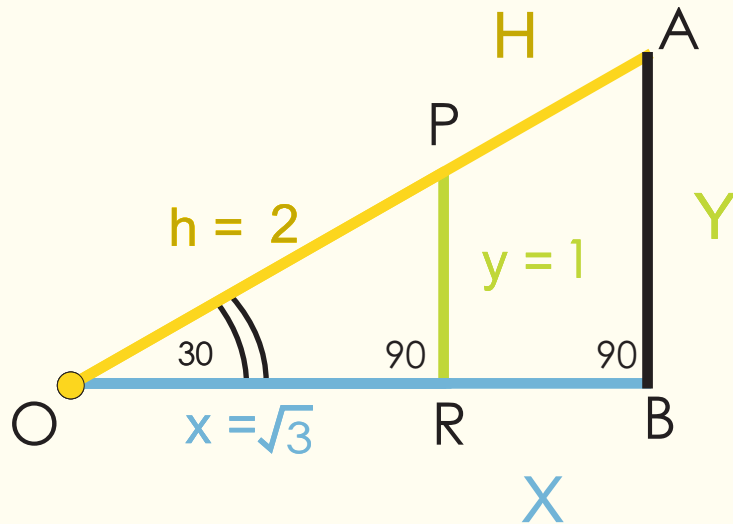
Let's review what we know about the trigonometric measures of an angle of 30 degrees.

Similar Triangles

Figures are said to be similar if they have the same shape but not necessarily the same size.

$$H = \overline{OA}$$

$$X = \overline{OB}$$



$$\frac{1}{2} = \frac{y}{h} = \frac{\text{opp}}{\text{hyp}} = \sin(30^\circ)$$

$$\frac{\sqrt{3}}{2} = \frac{x}{h} = \frac{\text{adj}}{\text{hyp}} = \cos(30^\circ)$$

$$\frac{1}{\sqrt{3}} = \frac{y}{x} = \frac{\text{opp}}{\text{adj}} = \tan(30^\circ)$$

THE TRIGONOMETRY OF ANGLES

(triangle measurement of angles)

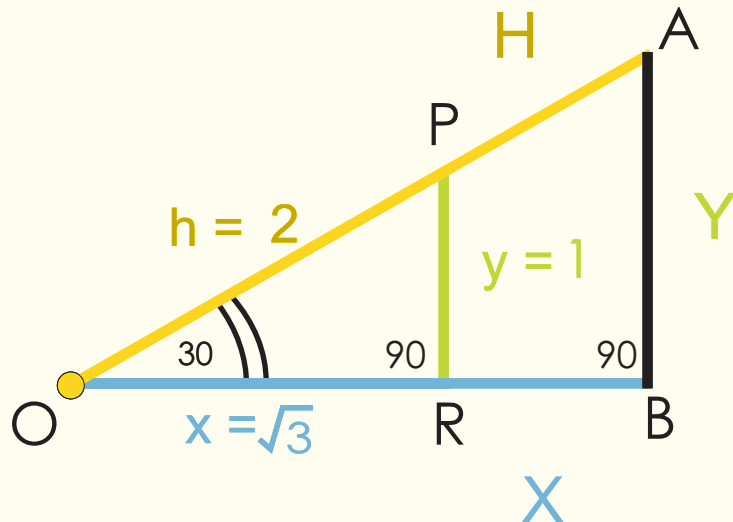
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THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
and Descartes (circa 1637)*

THE TRIGONOMETRY OF ANGLES

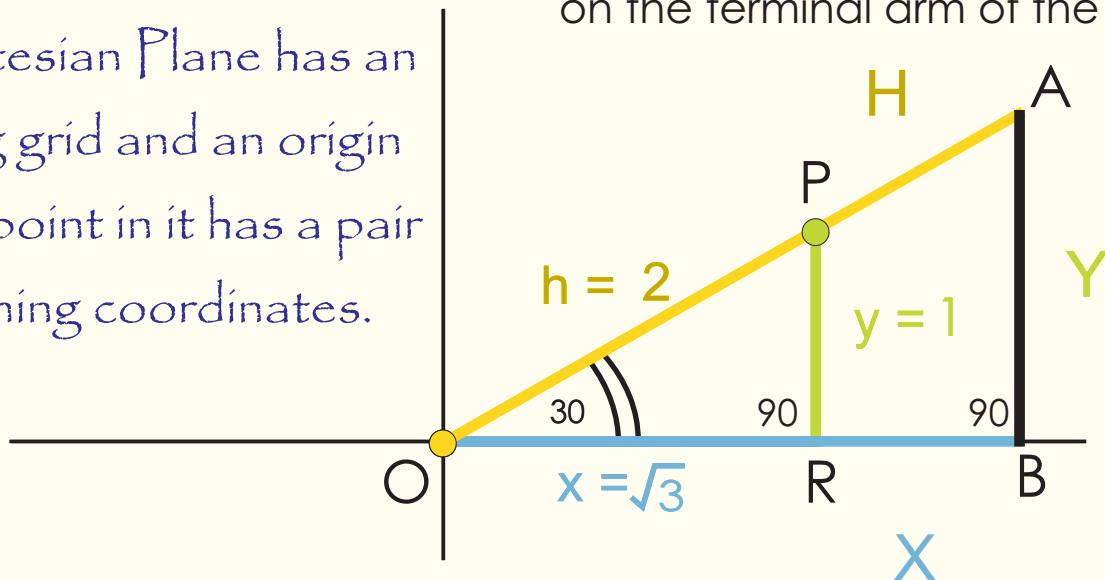
- *blending Pythagorus (circa 540 B.C.)
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Let's now put this setting
into the Cartesian Plane.

The Cartesian Plane has an
underlying grid and an origin
and each point in it has a pair
of positioning coordinates.

h = distance from P to the origin

$P = (x, y) = (\sqrt{3}, 1)$ is a point
on the terminal arm of the angle



$$\frac{1}{2} = \frac{y}{h} = \frac{\text{opp}}{\text{hyp}} = \sin(30^\circ)$$

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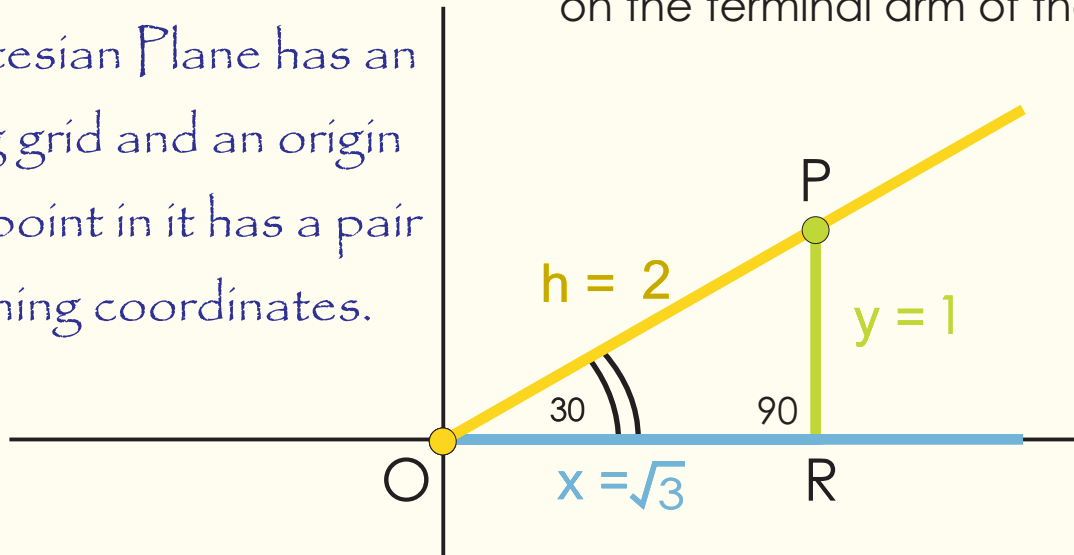
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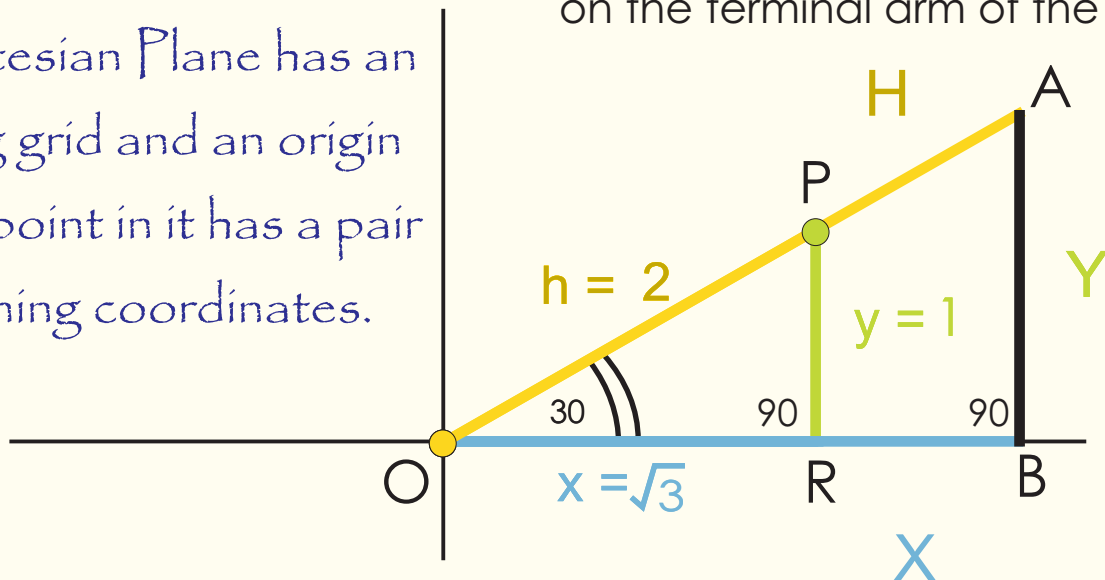
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THE TRIGONOMETRY OF ANGLES

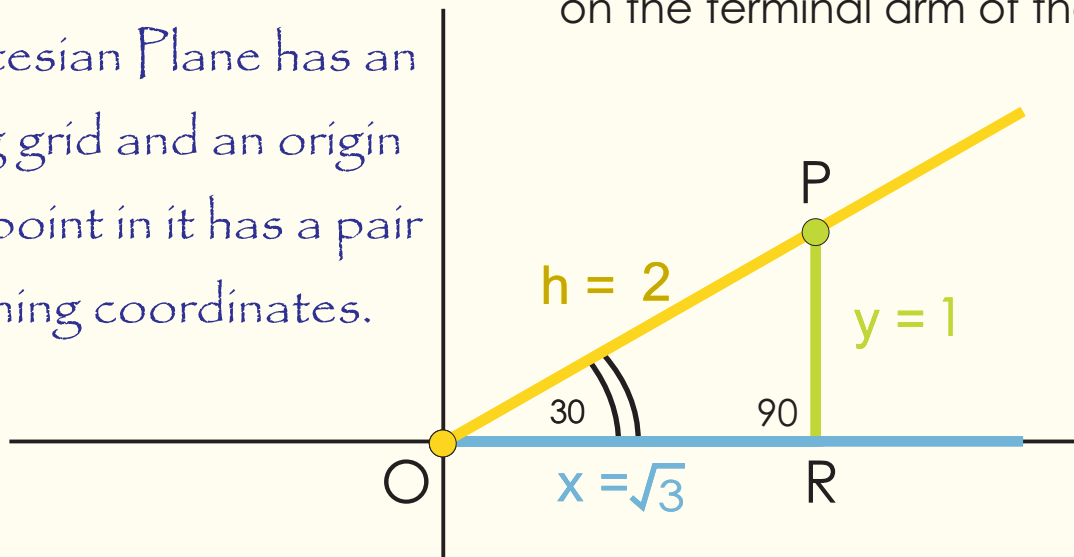
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THE TRIGONOMETRY OF ANGLES

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The Cartesian Plane has an underlying grid and an origin and each point in it has a pair of positioning coordinates.

Now we can talk about a standard position of an angle - meaning that it must have its vertex at the origin and its initial arm along the horizontal axis.

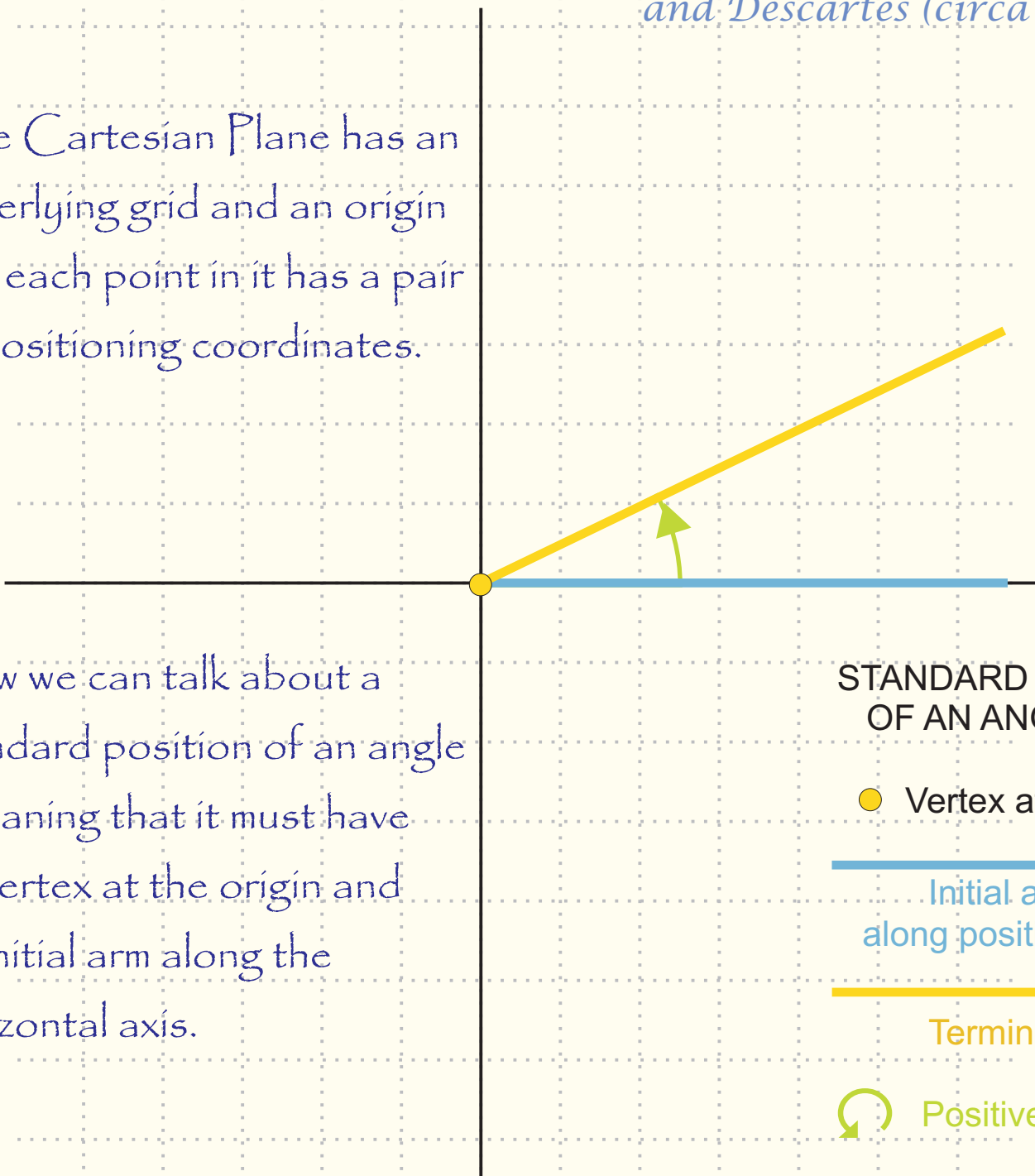
STANDARD POSITION OF AN ANGLE

● Vertex at origin

Initial arm
along positive x-axis

Terminal arm

↻ Positive angle

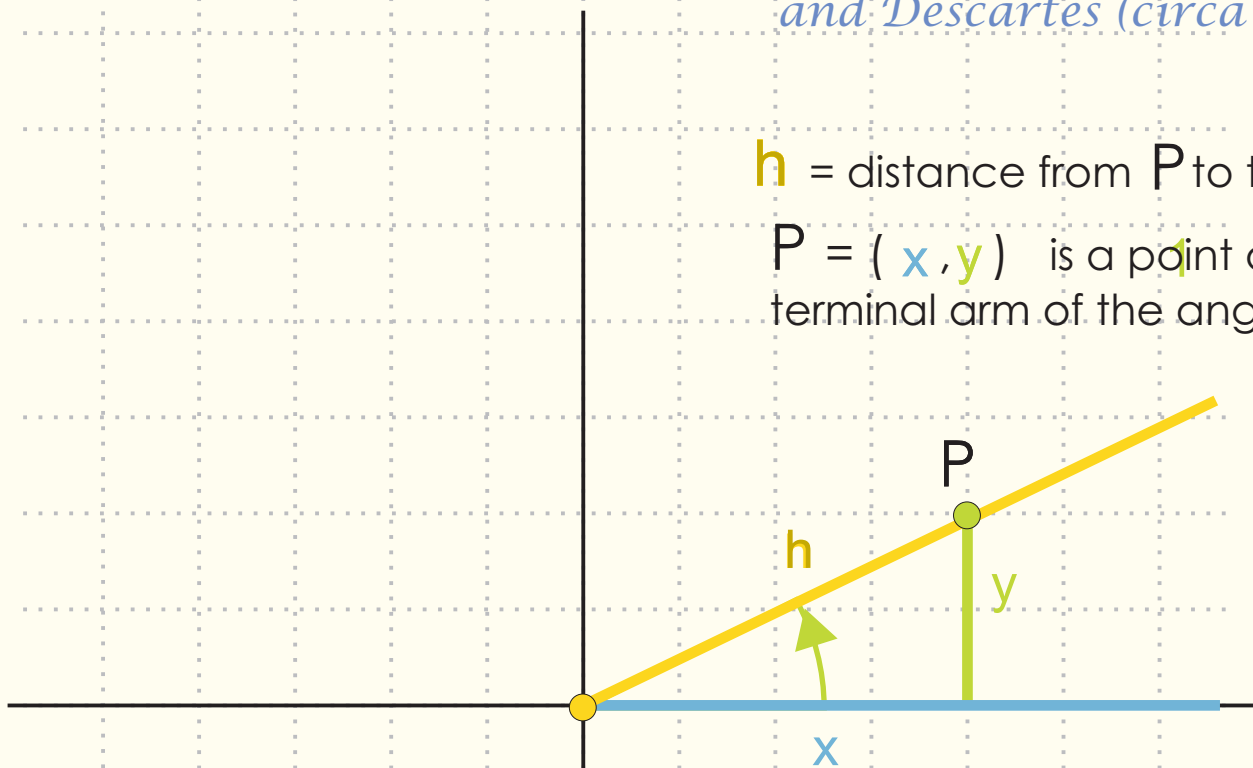


THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
and Descartes (circa 1637)*

h = distance from P to the origin

$P = (x, y)$ is a point on the terminal arm of the angle



Points on the terminal arm can be given Cartesian coordinates.

The horizontal and vertical coordinate values provide a way of constructing a right triangle which contains the angle at one of its vertices.

STANDARD POSITION
OF AN ANGLE

● Vertex at origin

Initial arm
along positive x-axis

Terminal arm

↻ Positive angle

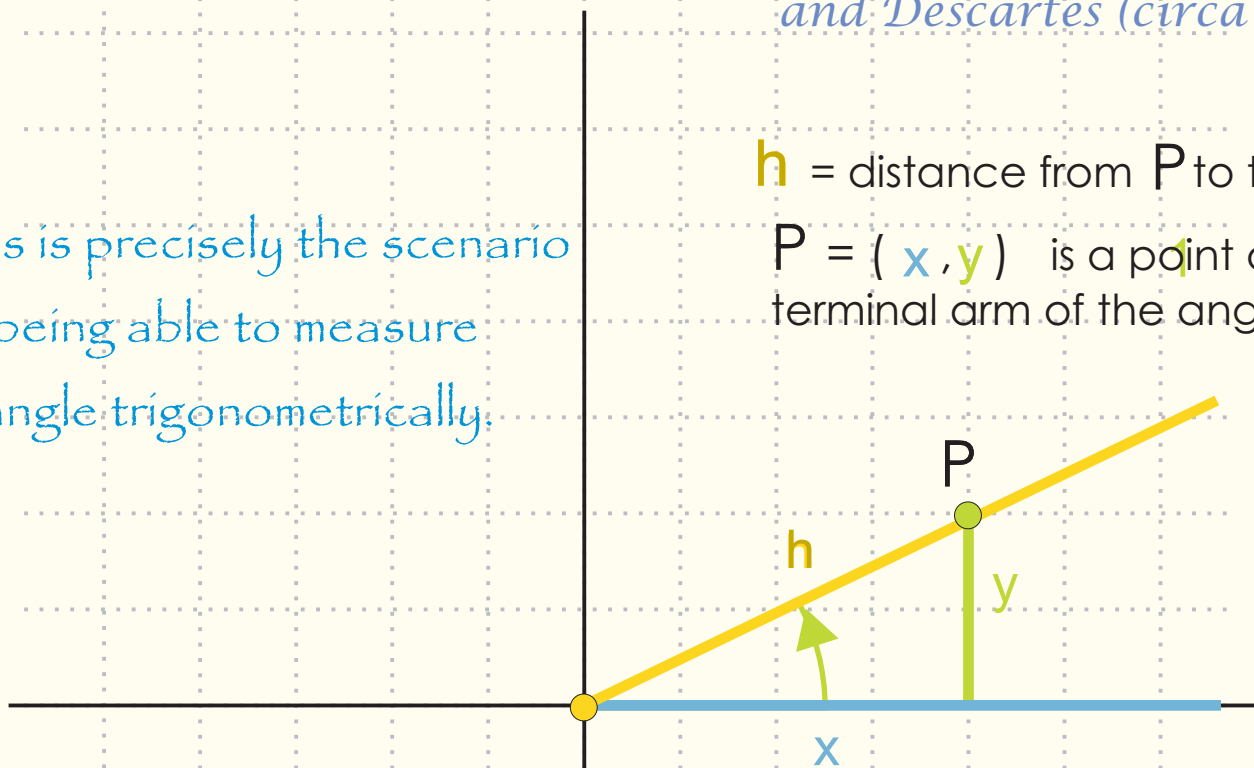
THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
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This is precisely the scenario
for being able to measure
an angle trigonometrically.

h = distance from P to the origin

$P = (x, y)$ is a point on the
terminal arm of the angle



Points on the terminal arm can
be given Cartesian coordinates.

The horizontal and vertical
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STANDARD POSITION
OF AN ANGLE

● Vertex at origin

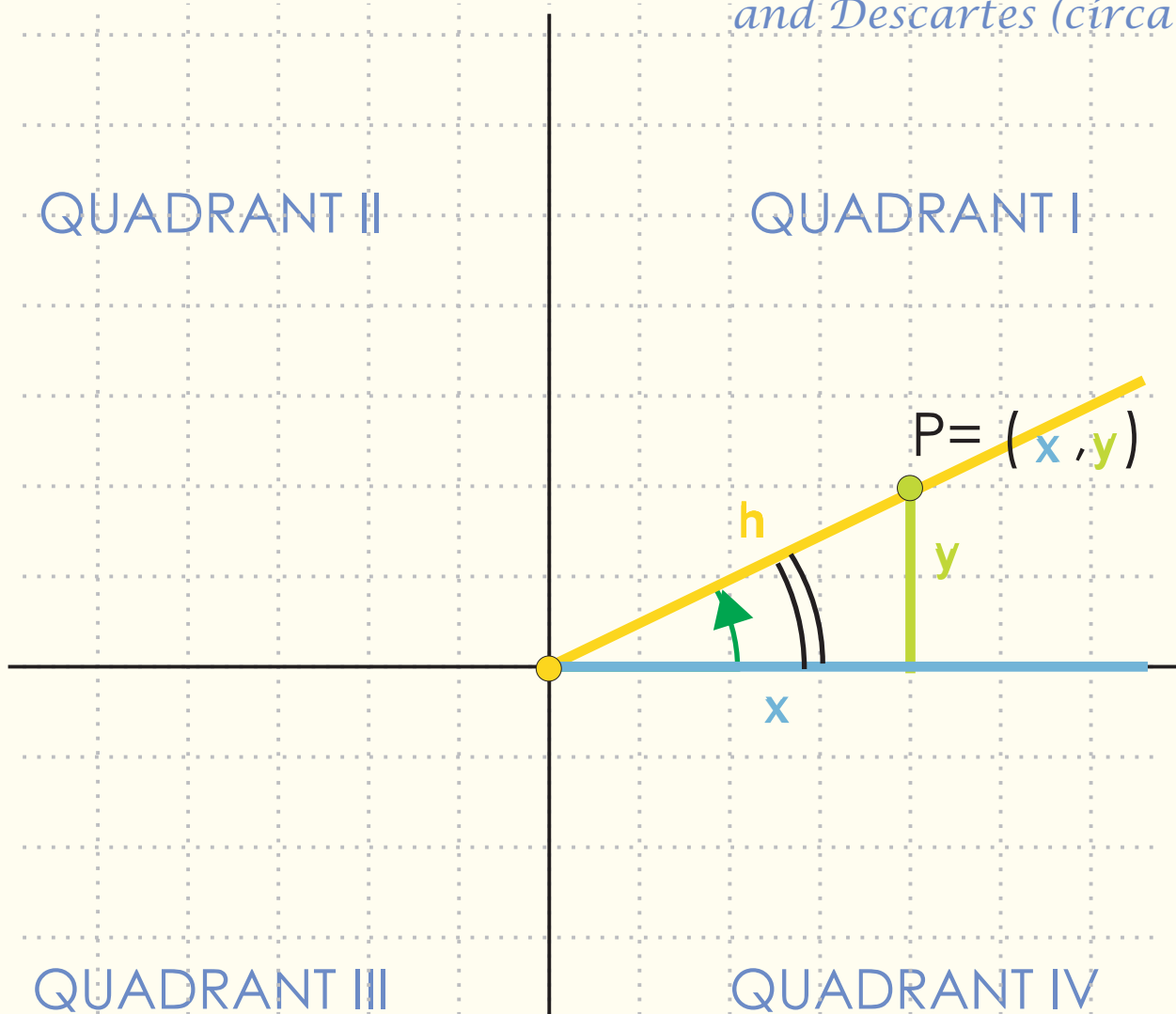
Initial arm
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Terminal arm

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THE TRIGONOMETRY OF ANGLES

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Because the origin lies in the centre of the Cartesian Plane at the intersection of the two perpendicular axes, the plane is subdivided into four different quadrants.

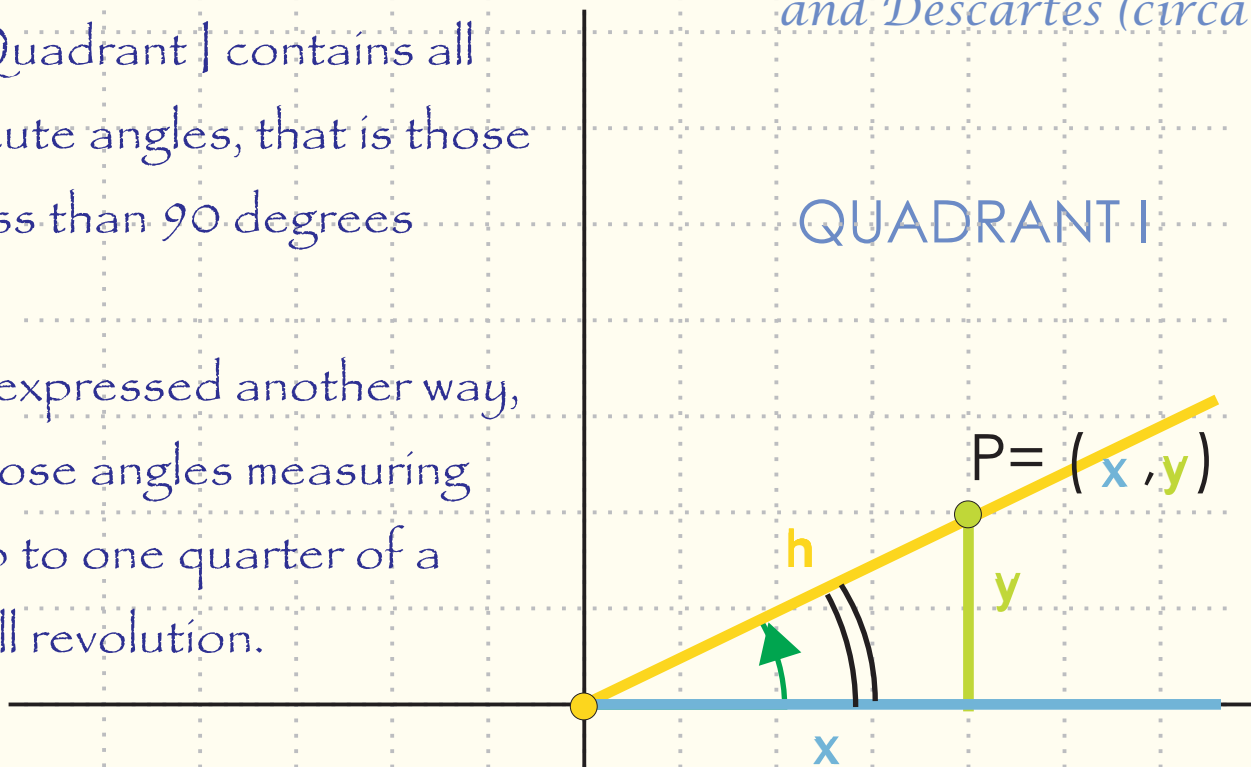
THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
and Descartes (circa 1637)*

Quadrant I contains all
acute angles, that is those
less than 90 degrees

- expressed another way,
those angles measuring
up to one quarter of a
full revolution.

QUADRANT I



Because the origin lies in the centre of the Cartesian Plane
at the intersection of the two perpendicular axes, the plane
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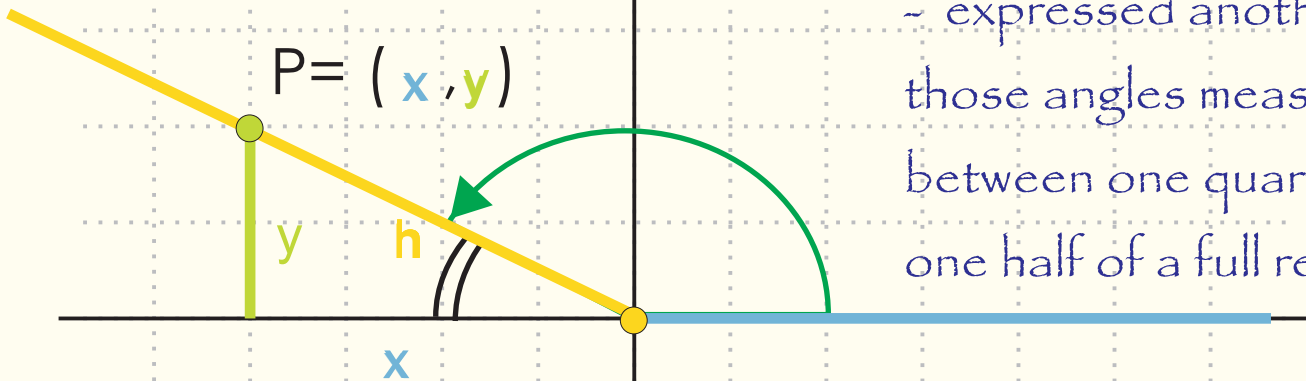
THE TRIGONOMETRY OF ANGLES

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QUADRANT II

Quadrant II contains all obtuse angles, that is those between 90 and 180 degrees

~ expressed another way, those angles measuring between one quarter and one half of a full revolution.



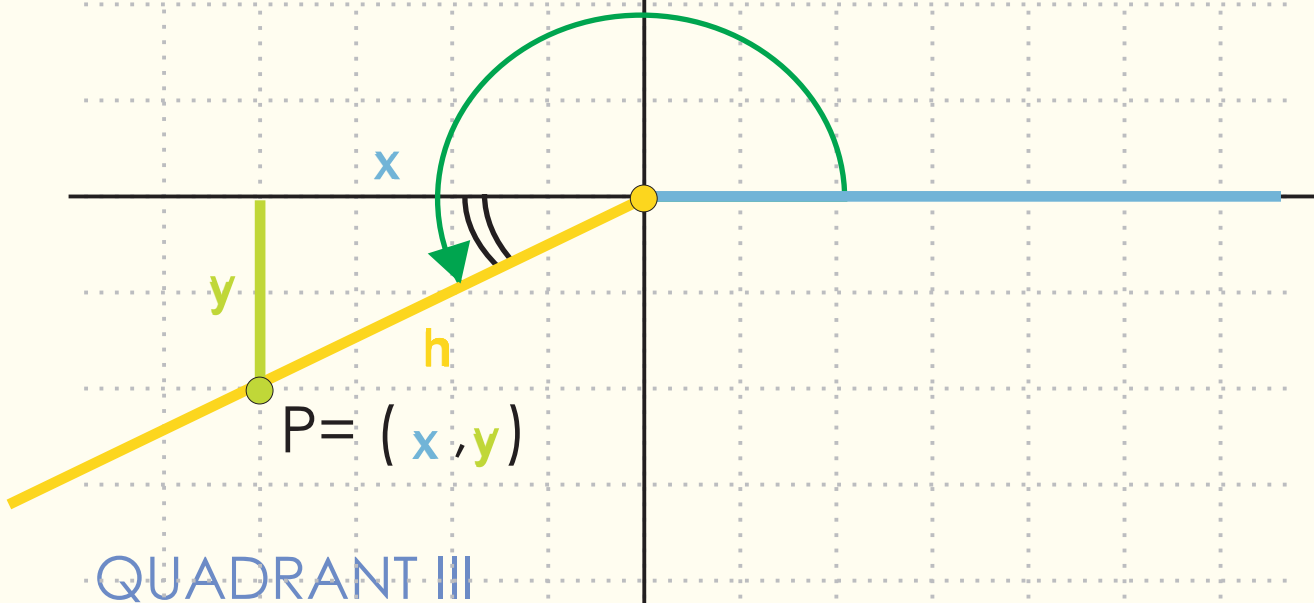
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Quadrant III contains all
standard position angles
which are between
180 and 270 degrees

- that is, those angles between
one half and three quarters
of a full revolution.



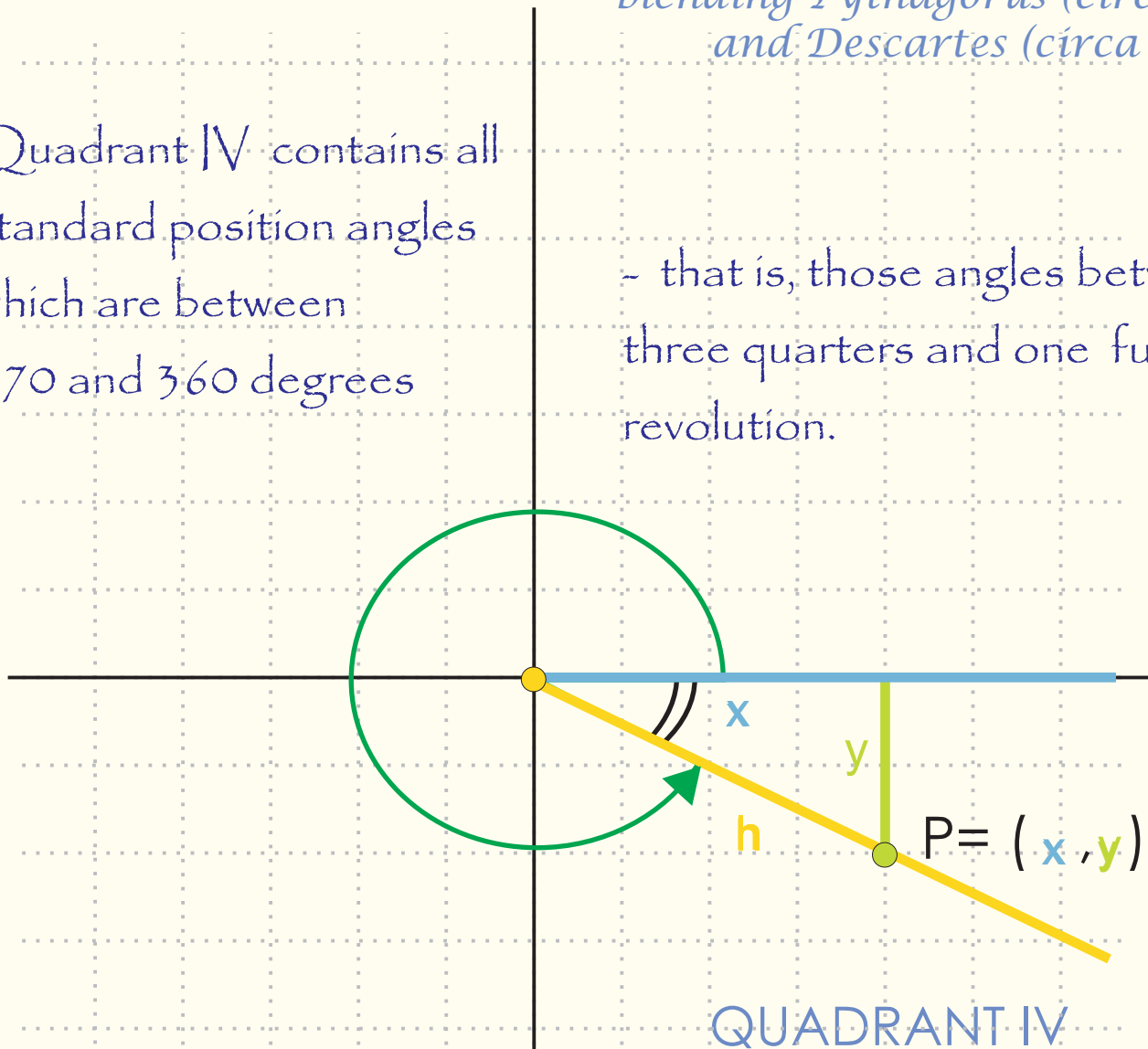
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THE TRIGONOMETRY OF ANGLES

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Quadrant IV contains all
standard position angles
which are between
270 and 360 degrees

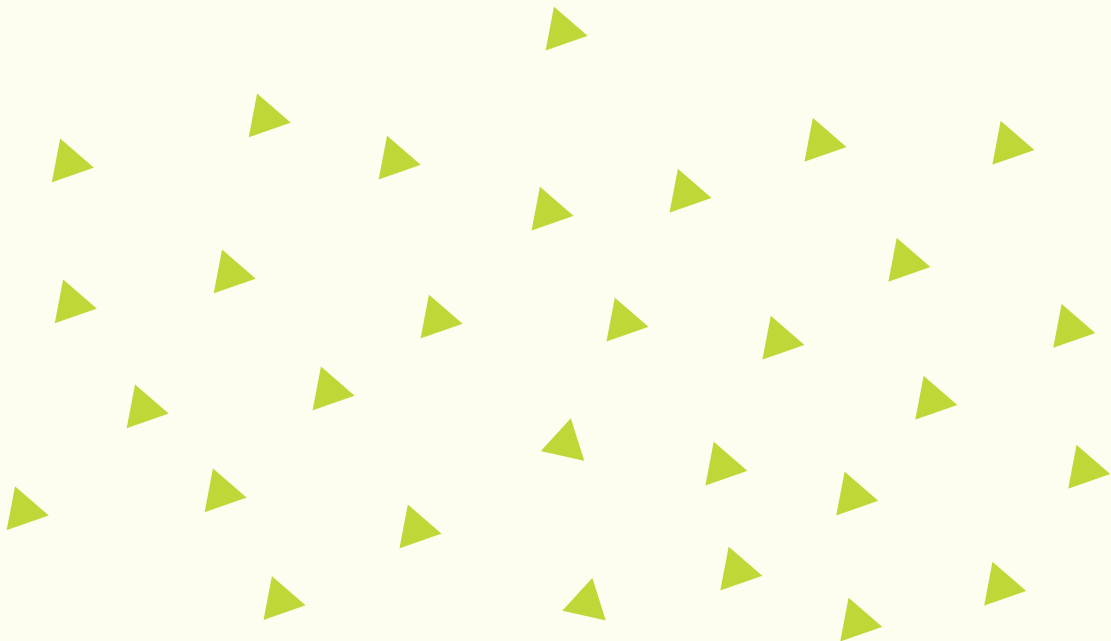
- that is, those angles between
three quarters and one full
revolution.



Because the origin lies in the centre of the Cartesian Plane
at the intersection of the two perpendicular axes, the plane
is subdivided into four different quadrants.

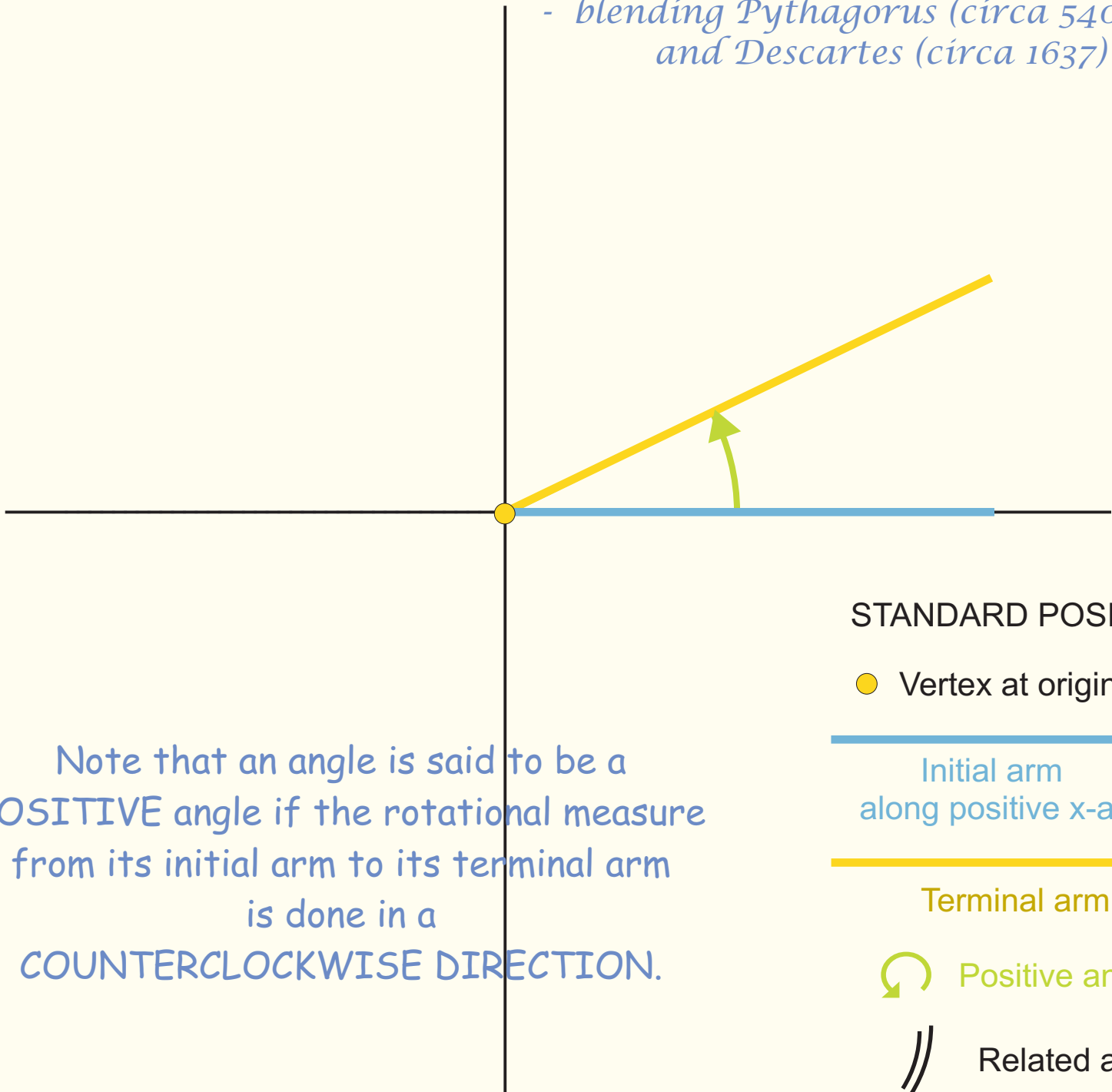
THE TRIGONOMETRY OF ANGLES

- generalized to all positive angles



THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
and Descartes (circa 1637)*



STANDARD POSITION

● Vertex at origin

Initial arm
along positive x-axis

Terminal arm

↻ Positive angle

// Related angle

Note that an angle is said to be a
POSITIVE angle if the rotational measure
from its initial arm to its terminal arm
is done in a
COUNTERCLOCKWISE DIRECTION.

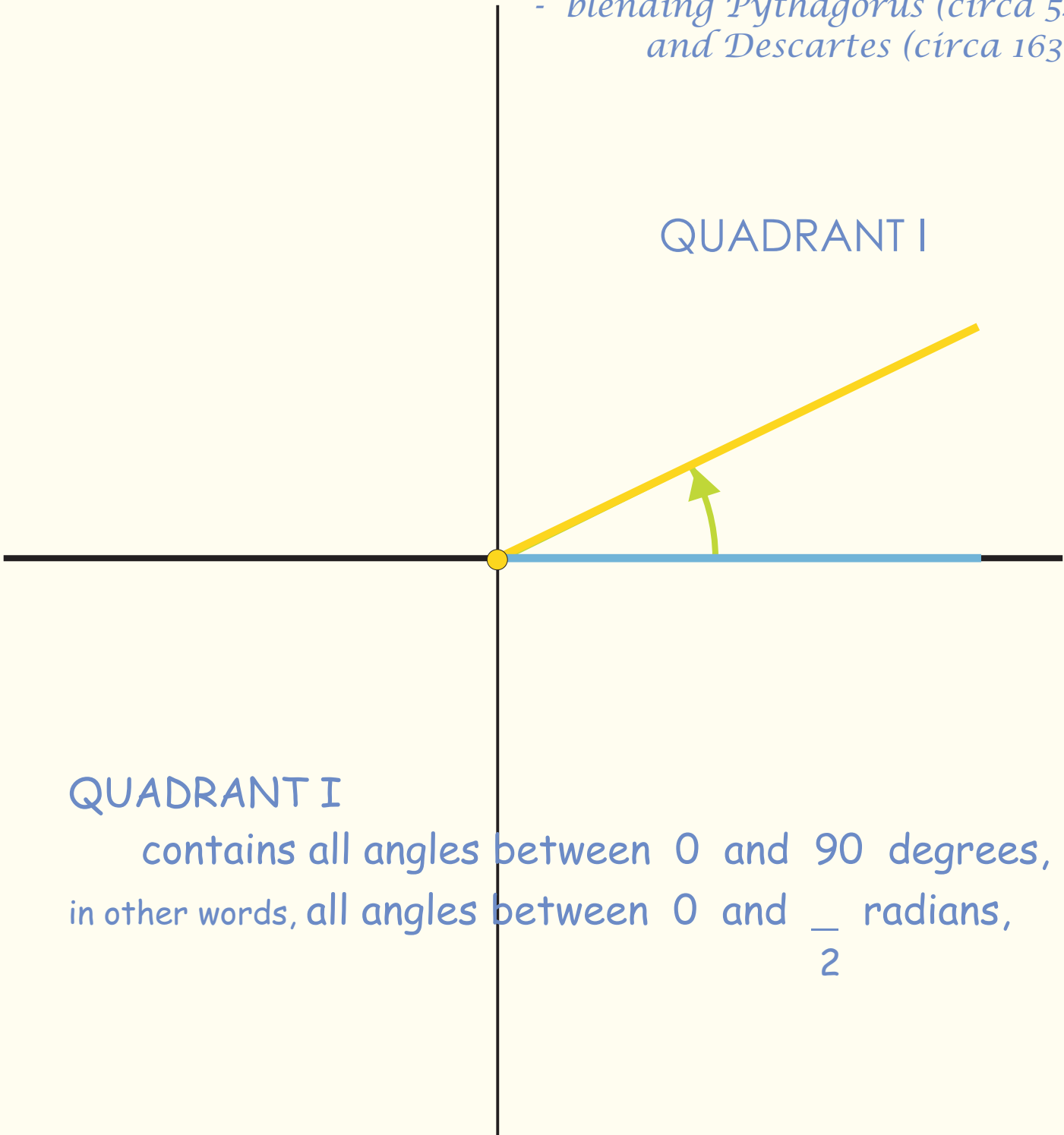
THE TRIGONOMETRY OF
FIRST QUADRANT ANGLES

QUADRANT I

THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
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QUADRANT I

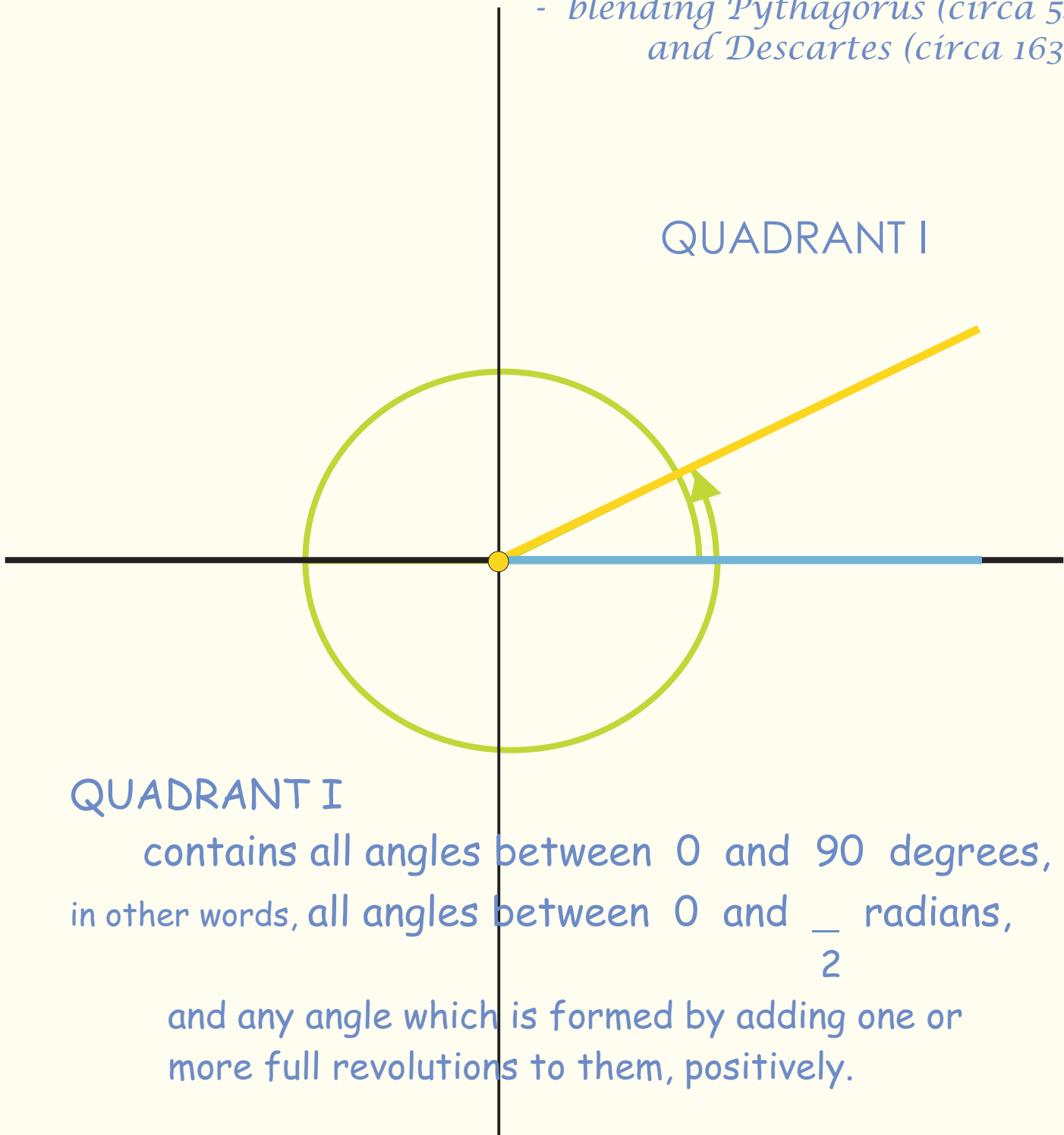


QUADRANT I

contains all angles between 0 and 90 degrees,
in other words, all angles between 0 and $\frac{\pi}{2}$ radians,

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QUADRANT I

QUADRANT I

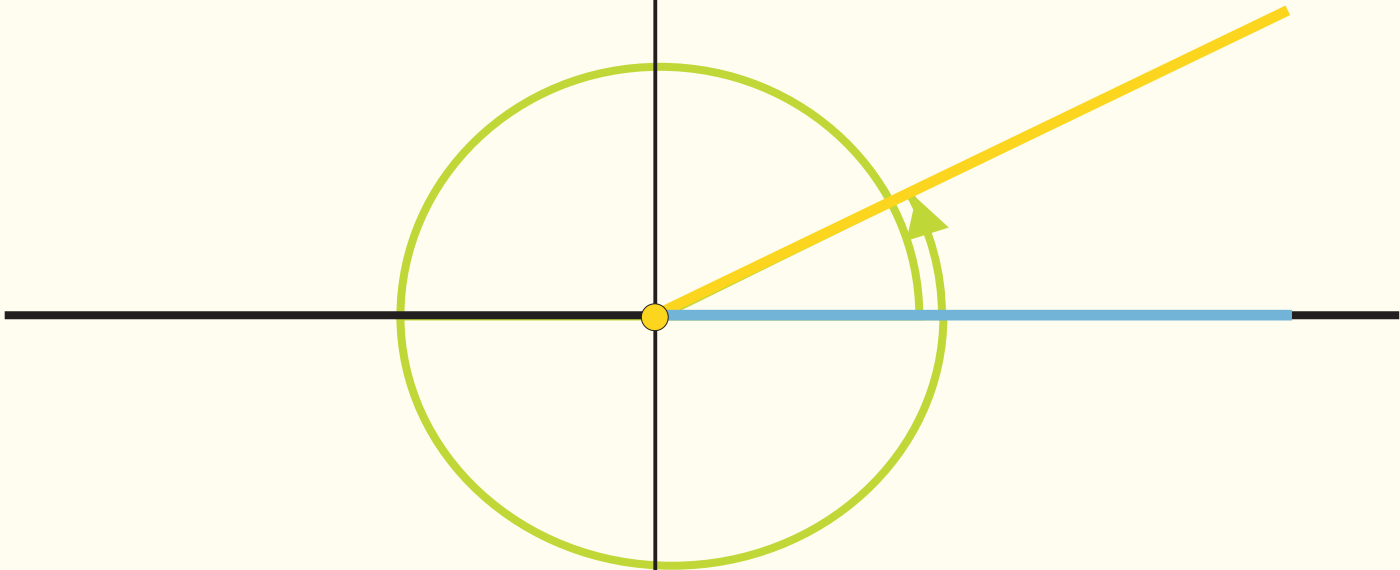
contains all angles between 0 and 90 degrees,
in other words, all angles between 0 and $\frac{\pi}{2}$ radians,

and any angle which is formed by adding one or
more full revolutions to them, positively.

THE TRIGONOMETRY OF ANGLES

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QUADRANT I



Quadrant I angles include:

$$30 + 360 = 390 \text{ degrees, or } \frac{\quad}{6} + 2 = \frac{13}{6} \text{ radians,}$$

$$45 + 720 = 765 \text{ degrees, or } \frac{\quad}{4} + 4 = \frac{17}{4} \text{ radians,}$$

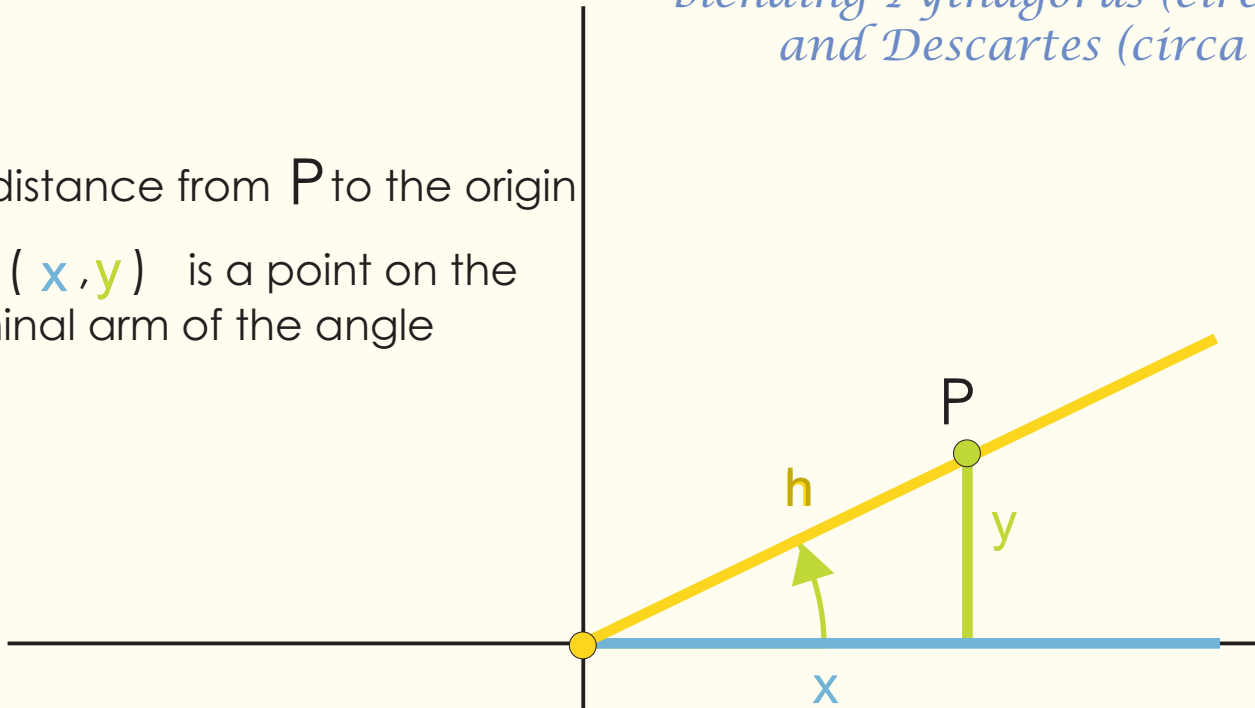
$$60 + 360 = 420 \text{ degrees, or } \frac{\quad}{3} + 2 = \frac{7}{3} \text{ radians.}$$

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h = distance from P to the origin

$P = (x, y)$ is a point on the
terminal arm of the angle



sine of an angle

$$\sin (\quad) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h}$$

cosine of an angle

$$\cos (\quad) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

tangent of an angle

$$\tan (\quad) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

STANDARD POSITION
OF AN ANGLE

● Vertex at origin

Initial arm
along positive x-axis

Terminal arm

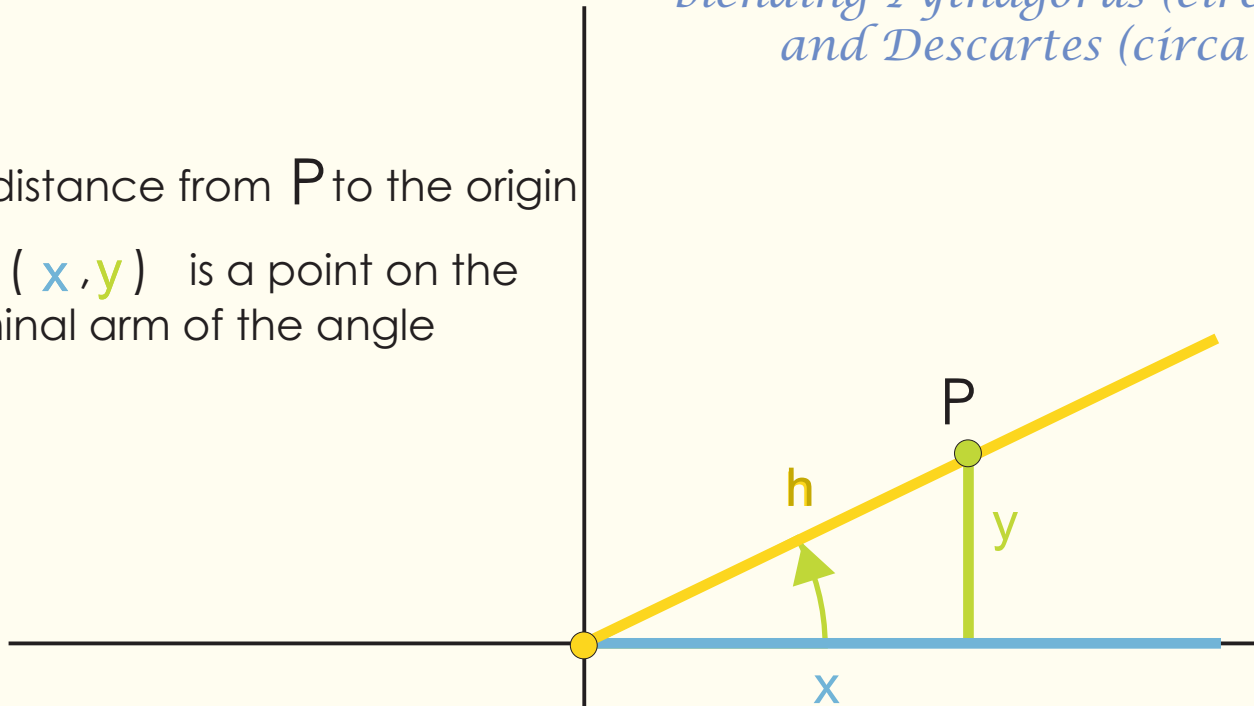
↻ Positive angle

THE TRIGONOMETRY OF ANGLES

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cosecant of an angle

$$\csc(\quad) := \frac{\text{hyp}}{\text{opp}} = \frac{h}{y}$$

cosine of an angle

$$\sec(\quad) := \frac{\text{hyp}}{\text{adj}} = \frac{h}{x}$$

cotangent of an angle

$$\cot(\quad) := \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$$

STANDARD POSITION
OF AN ANGLE

● Vertex at origin

Initial arm
along positive x-axis

Terminal arm

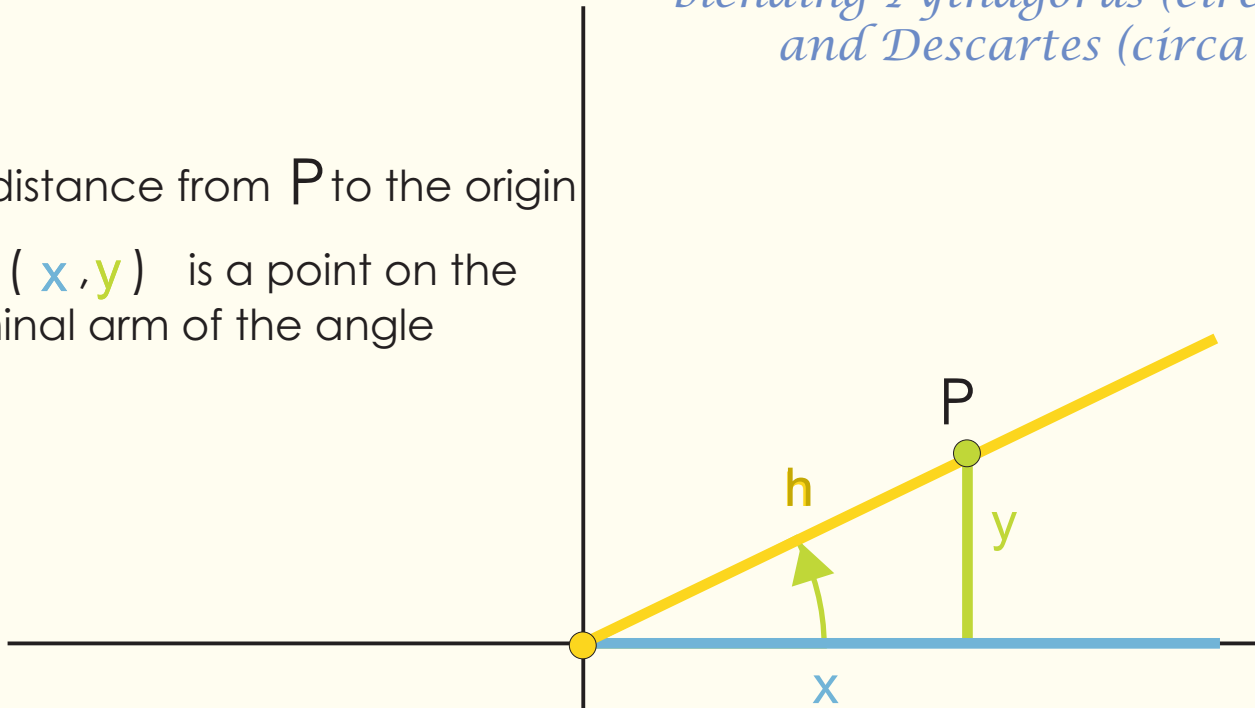
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cosine of an angle

$$\cos (\quad) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

tangent of an angle

$$\tan (\quad) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

cosecant of an angle

$$\csc (\quad) := \frac{\text{hyp}}{\text{opp}} = \frac{h}{y}$$

cosine of an angle

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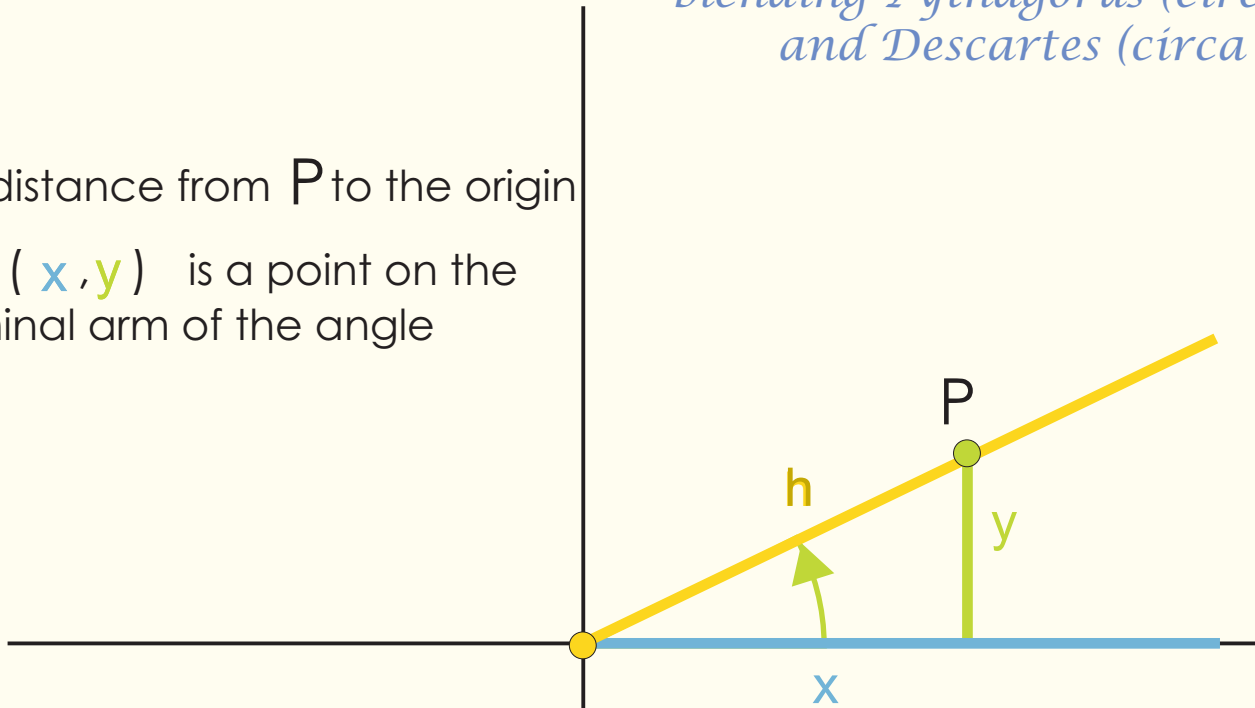
Note that: All SIX trigonometric functions of
any angle in the first quadrant have positive values.

THE TRIGONOMETRY OF ANGLES

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h = distance from **P** to the origin

P = (**x**, **y**) is a point on the
terminal arm of the angle



sine of an angle

$$\sin (\quad) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h}$$

cosine of an angle

$$\cos (\quad) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

tangent of an angle

$$\tan (\quad) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

The trigonometric measures
of an angle in the first
quadrant are measured using
the sides of the triangle in
the first quadrant containing
the angle itself.

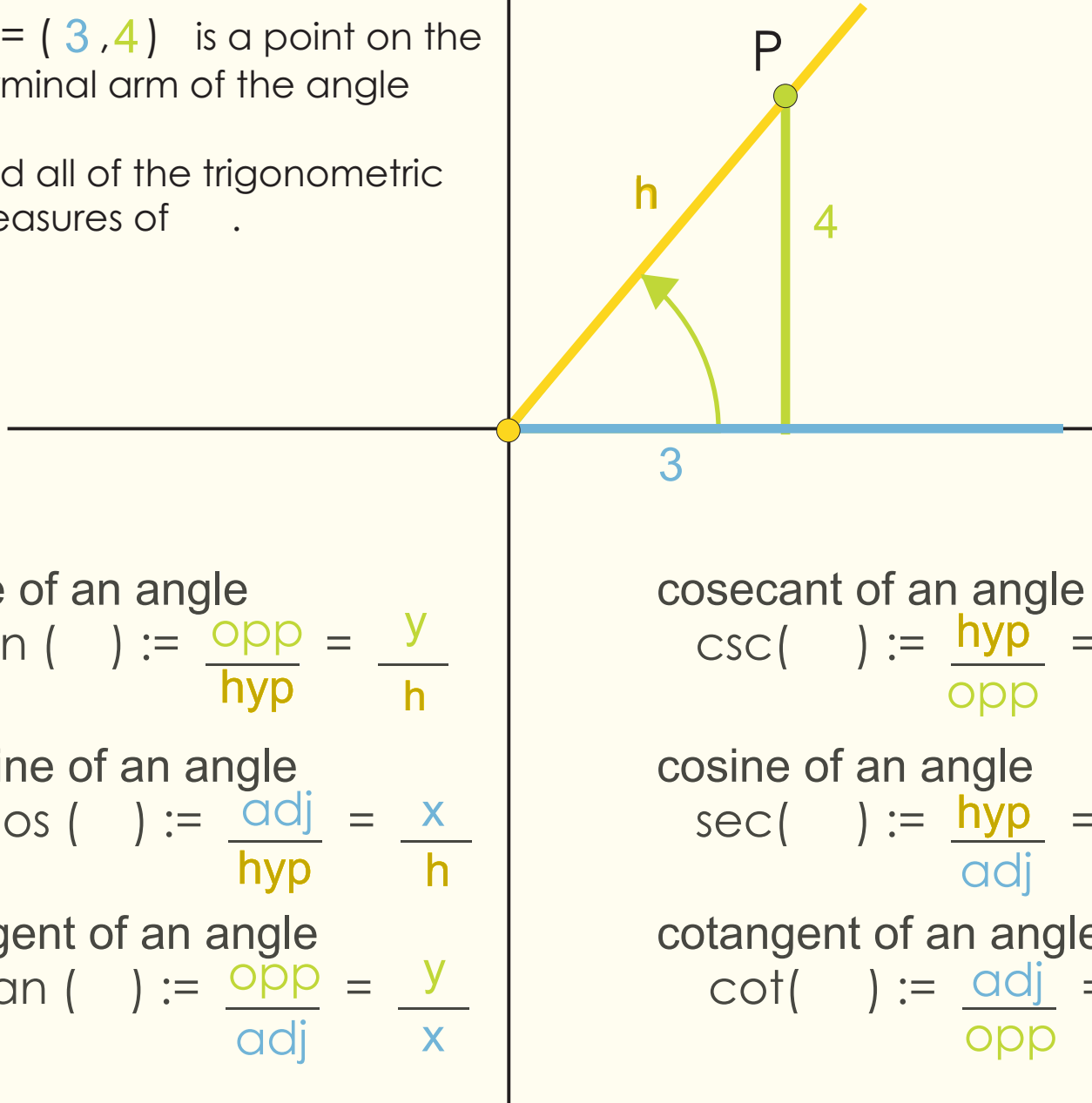
THE TRIGONOMETRY OF ANGLES

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EXAMPLE 1

$P = (3, 4)$ is a point on the terminal arm of the angle

Find all of the trigonometric measures of .



sine of an angle

$$\sin (\quad) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h}$$

cosine of an angle

$$\cos (\quad) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

tangent of an angle

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cotangent of an angle

$$\cot (\quad) := \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$$

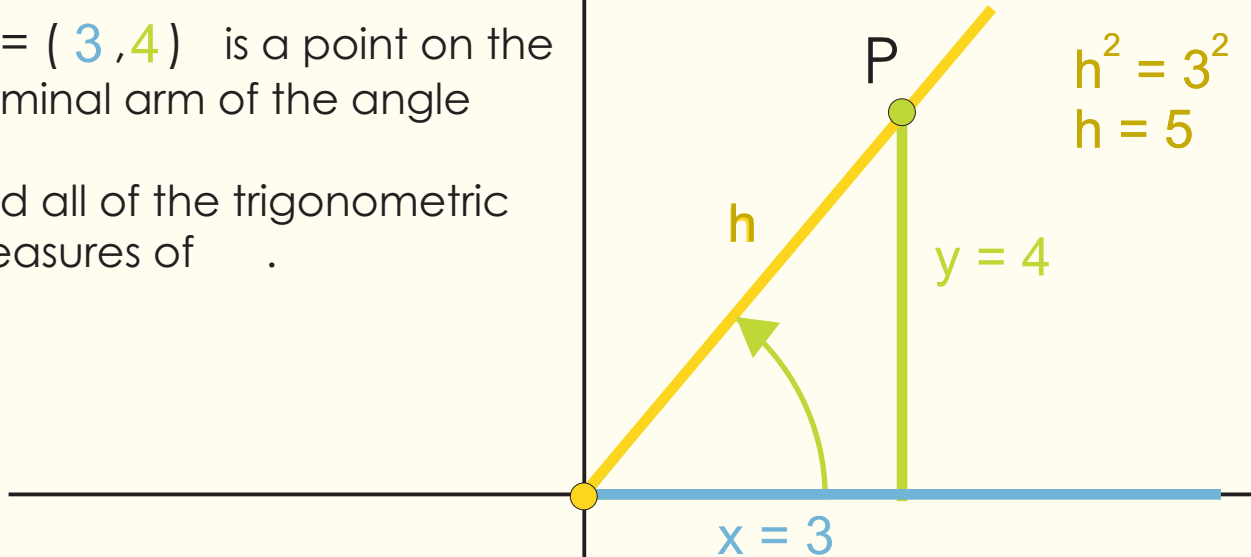
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EXAMPLE 1

$P = (3, 4)$ is a point on the terminal arm of the angle

Find all of the trigonometric measures of .



$$h^2 = 3^2 + 4^2 \text{ or } h = 5$$

sine of an angle

$$\sin (\quad) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h}$$

cosine of an angle

$$\cos (\quad) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

tangent of an angle

$$\tan (\quad) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

cosecant of an angle

$$\csc (\quad) := \frac{\text{hyp}}{\text{opp}} = \frac{h}{y}$$

cosine of an angle

$$\sec (\quad) := \frac{\text{hyp}}{\text{adj}} = \frac{h}{x}$$

cotangent of an angle

$$\cot (\quad) := \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$$

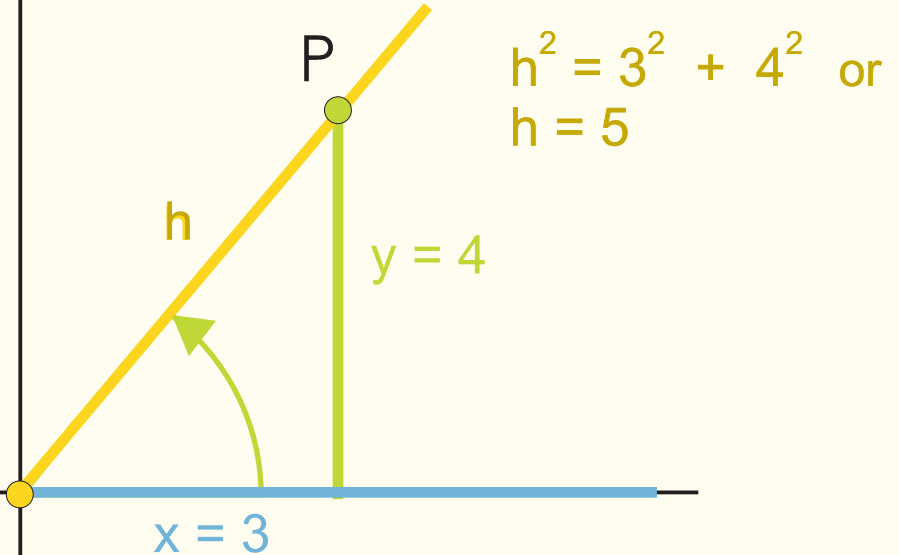
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EXAMPLE 1

$P = (3, 4)$ is a point on the terminal arm of the angle

Find all of the trigonometric measures of .



$$h^2 = 3^2 + 4^2 \text{ or } h = 5$$

Answer:

sine of an angle

$$\sin () := \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$$

cosine of an angle

$$\cos () := \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$$

tangent of an angle

$$\tan () := \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$$

cosecant of an angle

$$\csc () := \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

cosine of an angle

$$\sec () := \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

cotangent of an angle

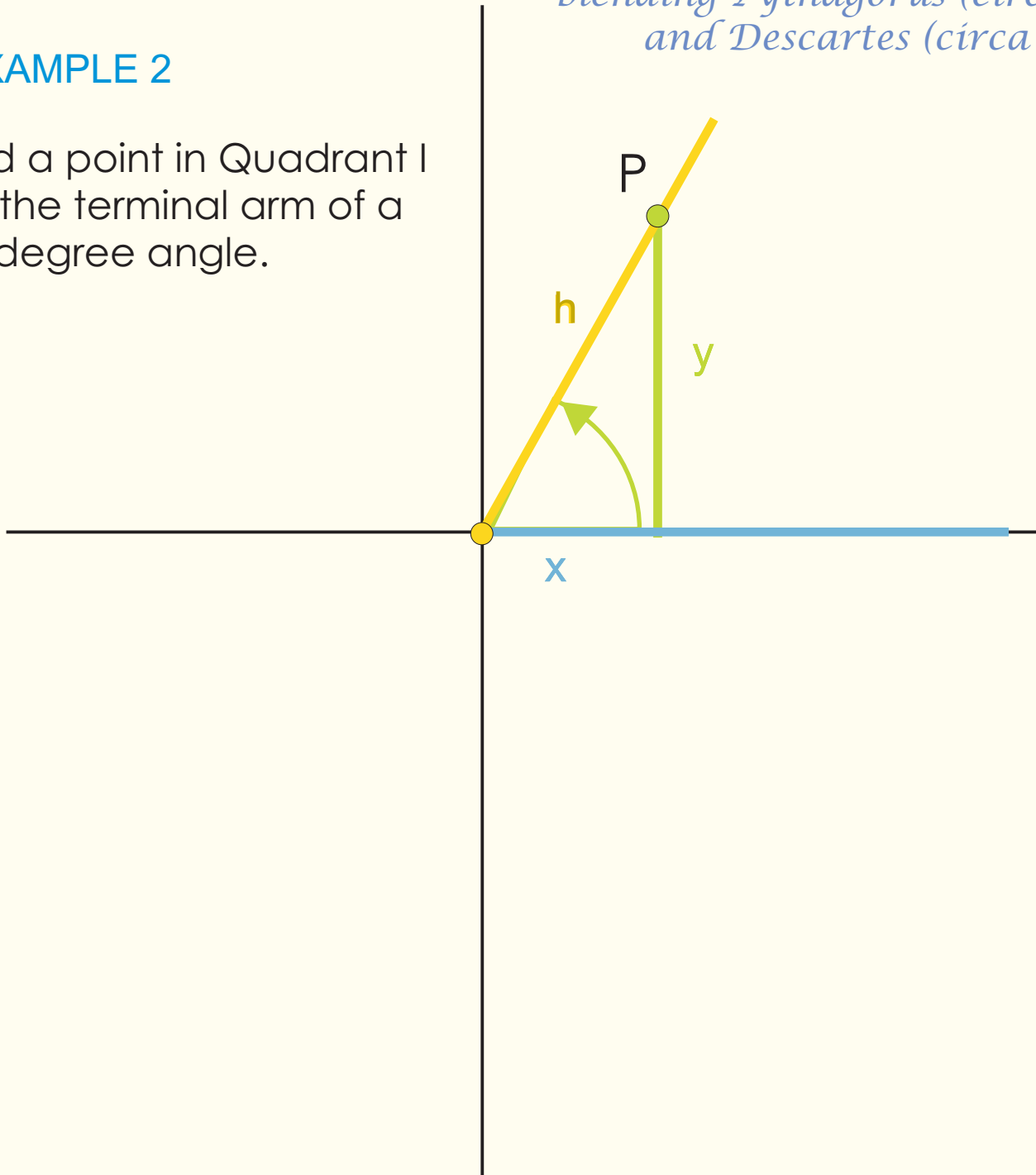
$$\cot () := \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

THE TRIGONOMETRY OF ANGLES

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EXAMPLE 2

Find a point in Quadrant I
on the terminal arm of a
60 degree angle.

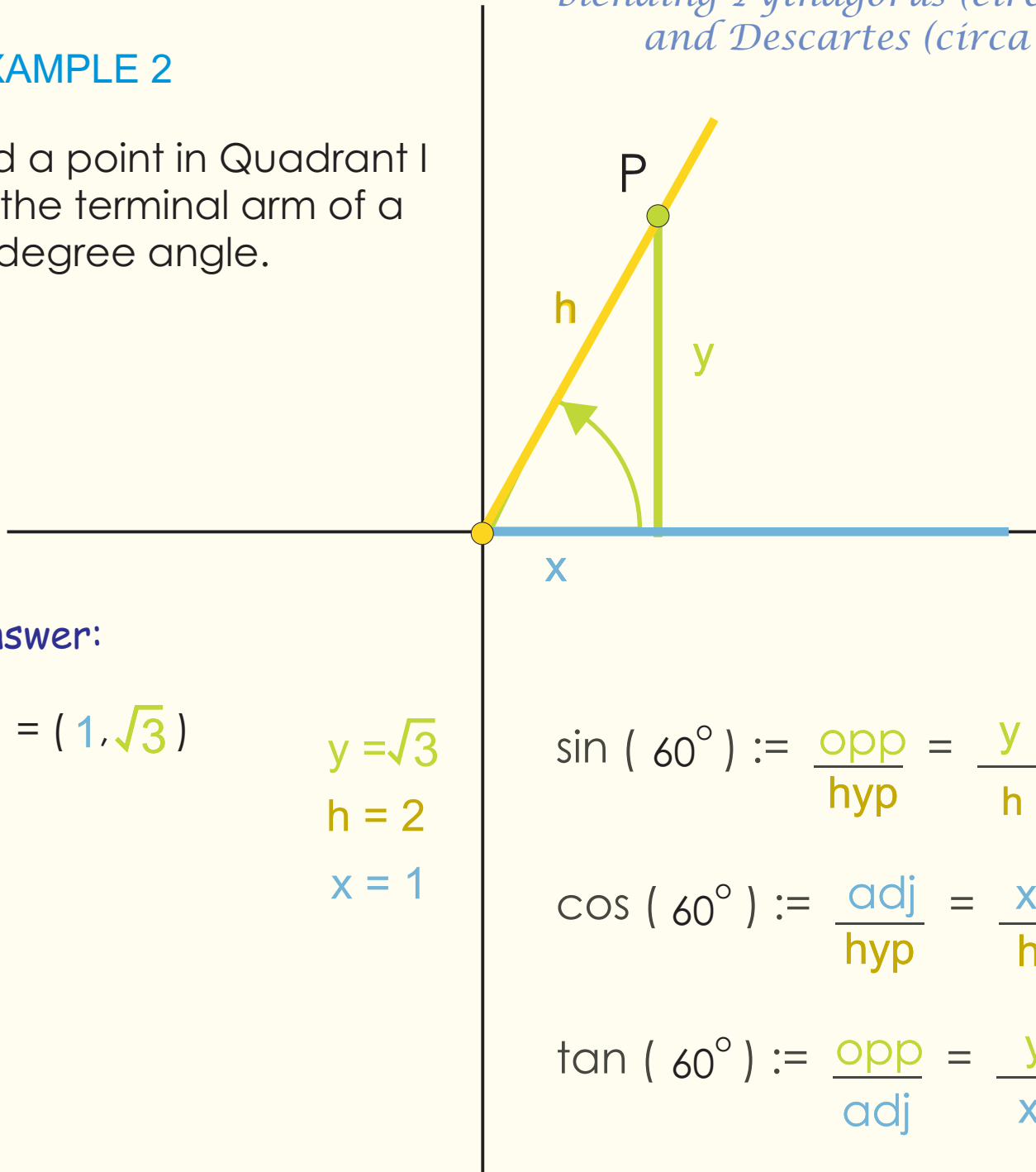


THE TRIGONOMETRY OF ANGLES

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EXAMPLE 2

Find a point in Quadrant I
on the terminal arm of a
60 degree angle.



Answer:

$$P = (1, \sqrt{3})$$

$$y = \sqrt{3}$$

$$h = 2$$

$$x = 1$$

$$\sin (60^\circ) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h} = \frac{\sqrt{3}}{2}$$

$$\cos (60^\circ) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h} = \frac{1}{2}$$

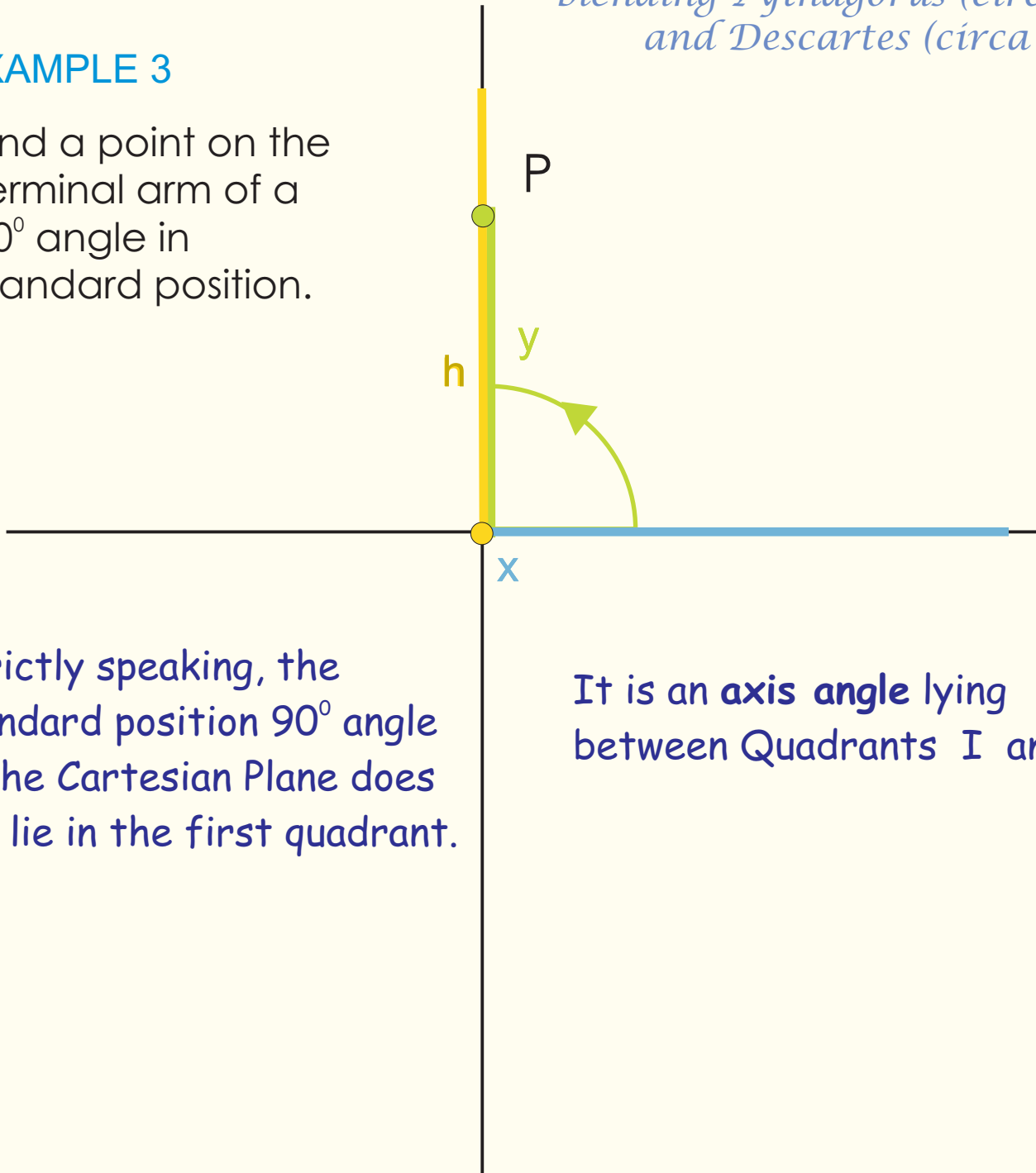
$$\tan (60^\circ) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{\sqrt{3}}{1}$$

THE TRIGONOMETRY OF ANGLES

- *blending Pythagorus (circa 540 B.C.)
and Descartes (circa 1637)*

EXAMPLE 3

Find a point on the
terminal arm of a
 90° angle in
standard position.



Strictly speaking, the
standard position 90° angle
in the Cartesian Plane does
not lie in the first quadrant.

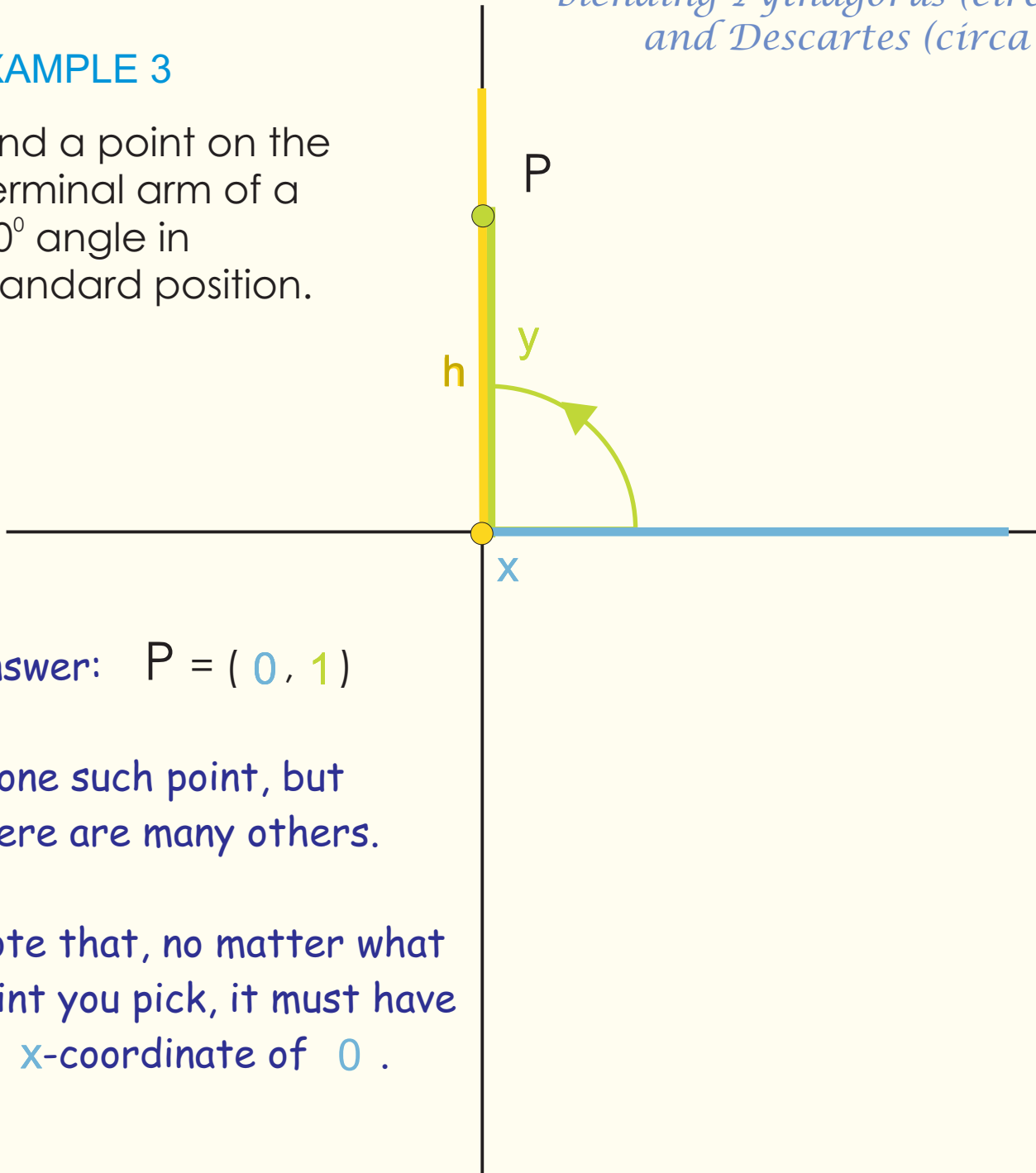
It is an **axis angle** lying
between Quadrants I and II.

THE TRIGONOMETRY OF ANGLES

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EXAMPLE 3

Find a point on the terminal arm of a 90° angle in standard position.



Answer: $P = (0, 1)$

is one such point, but
there are many others.

Note that, no matter what
point you pick, it must have
an x-coordinate of 0 .

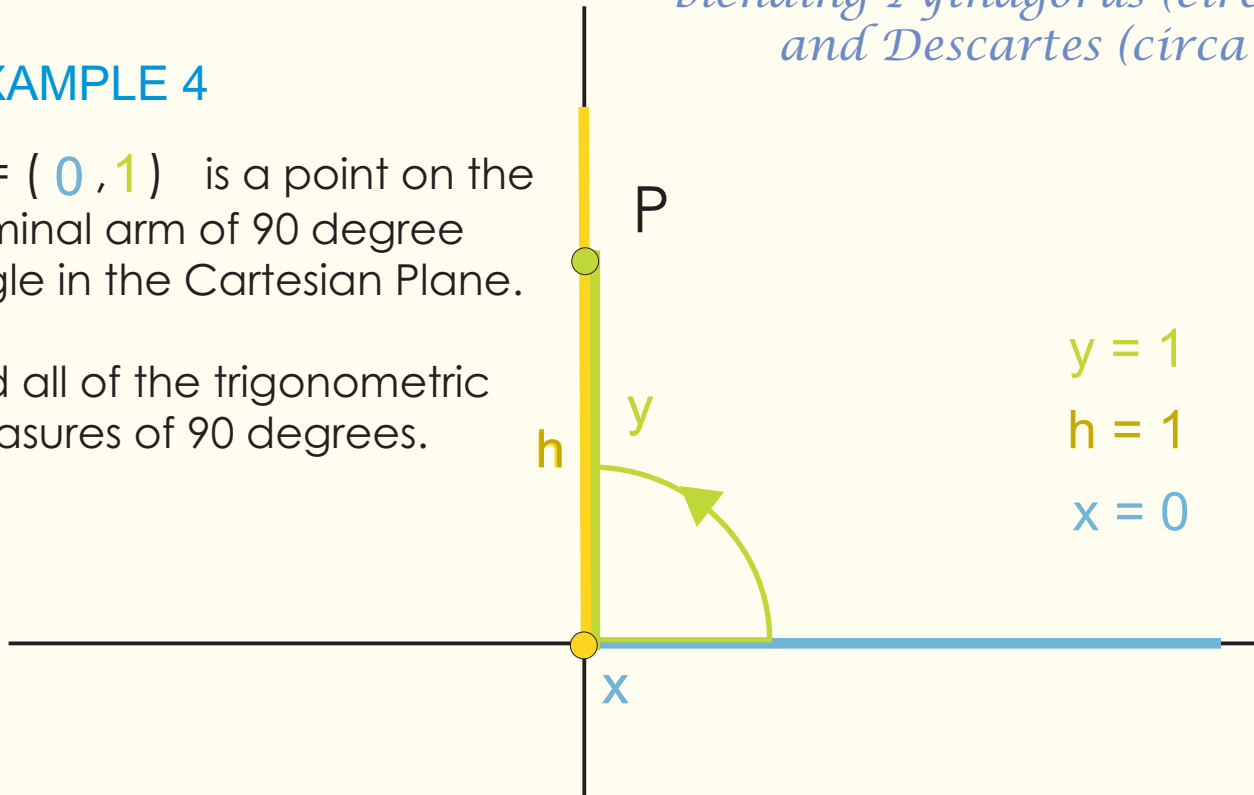
THE TRIGONOMETRY OF ANGLES

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EXAMPLE 4

$P = (0, 1)$ is a point on the terminal arm of 90 degree angle in the Cartesian Plane.

Find all of the trigonometric measures of 90 degrees.



$$\sin (90^\circ) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h}$$

$$\cos (90^\circ) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h}$$

$$\tan (90^\circ) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

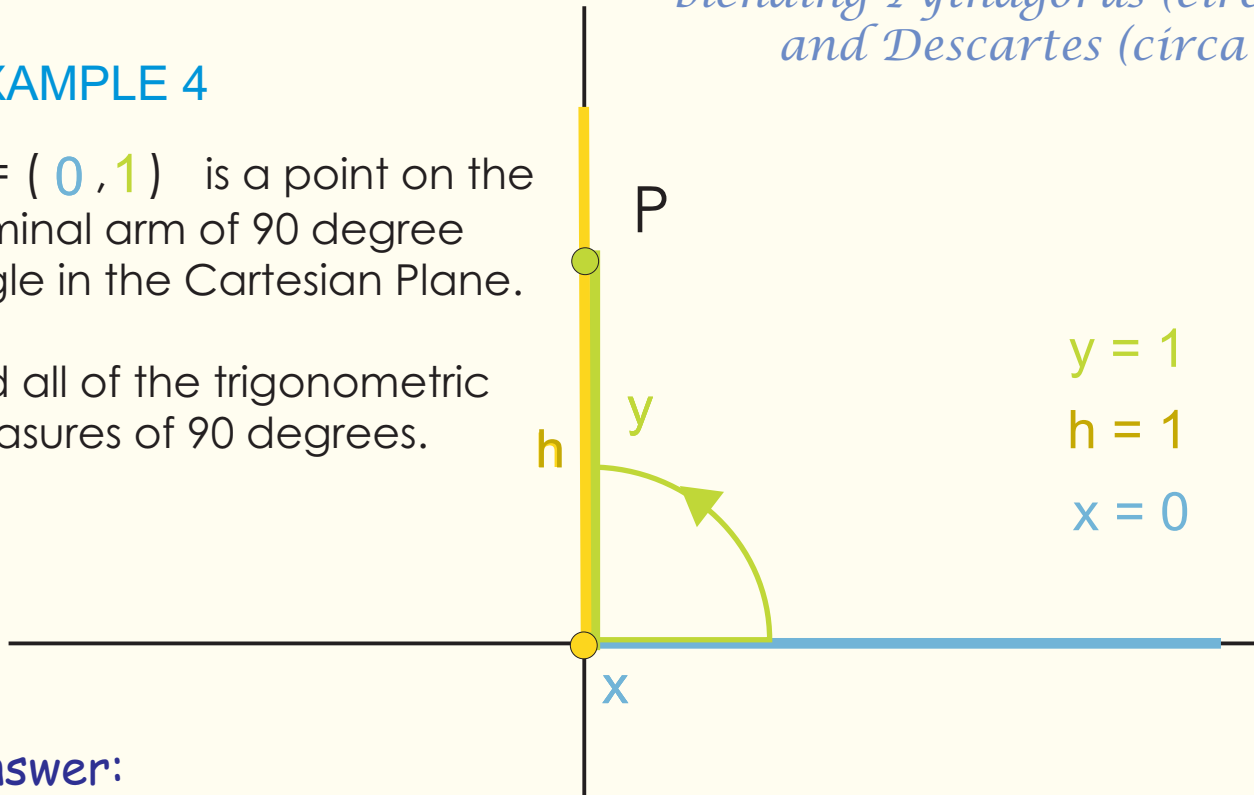
THE TRIGONOMETRY OF ANGLES

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EXAMPLE 4

$P = (0, 1)$ is a point on the terminal arm of 90 degree angle in the Cartesian Plane.

Find all of the trigonometric measures of 90 degrees.



Answer:

$$\sin (90^\circ) := \frac{\text{opp}}{\text{hyp}} = \frac{y}{h} = \frac{1}{1} = 1$$

$$\csc (90^\circ) = 1$$

$$\cos (90^\circ) := \frac{\text{adj}}{\text{hyp}} = \frac{x}{h} = \frac{0}{1} = 0$$

$$\sec (90^\circ) \text{ undefined}$$

$$\tan (90^\circ) := \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{1}{0} \text{ undefined}$$

$$\cot (90^\circ) = 0$$

QUADRANT II

QUADRANT I

$\sin (\)$ positive

$\cos (\)$ positive

$\tan (\)$ positive

QUADRANT III

QUADRANT IV



